Dynamics and Convergence Rate of Ordinal Comparison of Stochastic Discrete Event Systems

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Comportements dynamiques et convergence de la comparaison ordinale des systèmes stochatiques à événements discrets

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Résumé

Dans ce papier, nous considérons la comparaison ordinale lors de la simulation des systèmes stochastiques à événements discrets. Nous examinons les comportements dynamiques de la comparison ordinale dans un cadre général. Nous montrons que, pour certaines classes de systèmes, la probabilité d'obtenir une solution souhaitée à l'aide d'une approche ordinale converge de manière exponentielle alors que les variances des mesures de performance converge au mieux en $O(1/t^2)$, où t est le temps de simulation. Nous donnons également des arguments empiriques pour montrer que la convergence exponentielle existe pour des systèmes généraux.

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Abstract

This paper addresses the dynamics and the convergence of ordinal comparison in the simulation of stochastic discrete event systems. It examines properties of dynamic behaviours of ordinal comparison in a fairly general framework. Most importantly, it proves that for some important classes of discrete event systems, the probability of obtaining a desired solution using an ordinal comparison approach converges at exponential rate while the variances of the performance measures converge at best at rate $O(1/t^2)$, where t is the simulation time. Heuristic arguments are also provided to explain that exponential convergence rate holds for more general systems.

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1. Introduction

Optimization in discrete solution space becomes more and more important for discrete event dynamic systems. There are countless number of potential applications such as production capacity and buffer capacity dimensioning in manufacturing systems. The only general tool for evaluating such systems is the simulation. Due to the lack of viable optimization approaches, empirical and sometime blind solution search approaches are used.

A straightforward and widely used approach consists in: (i) simulating all candidate designs (i.e. all selected solutions) to obtain accurate enough estimation of the performance measures, and (ii) selecting the best design. The major problem with this approach is the requirement of long simulation run to obtain accurate enough performance estimators. It has been proved that, for most discrete event systems including Markov chains and regenerative processes, the variances of performance measures converge typically at rate $O(1/t)$ in time t. Convergence rate $O(1/t^2)$ can be obtained using sophisticated variance reduction techniques (see [17]). These convergence rates are usually unsatisfactory when the number of candidate designs is important.

The ordinal optimization approaches first proposed in [12] (see also [13] for an overview) reduce the computation burden by "combining some mind-set changes" concerning the problem of optimization of discrete event systems.

1. (Ranking) The primary concern of the ordinal optimization approaches is the ranking of the candidate solutions instead of their exact criterion values. Numerous simulations conducted by different authors for a wide range of problems have shown that the ranking stabilize before the convergence of the criterion value estimates.

2. (Parallel simulation) Ordinal optimization approaches generally simulate simultaneously different candidate solutions using common random variable generation or standard clock technique ([18]). It has been shown in [7, 9] that the resulted correlations between the criterion value estimation errors can only help and increase the chance of identifying good solutions very early in the simulation.

3. (Goal relaxation) To significantly reduce the simulation time, ordinal optimization approaches typically relax the goal of simulation to the isolation of a set of good candidate solutions. The observations of numerous simulation experiments indicate that it is possible to determine whether a candidate solution is good or bad very early in the simulation with high probability. Typical relaxing goal is to identify a small subset of candidate solutions containing at least one top-r solution with high probability. Estimates of this probability has been proposed in [3] for a class of discrete event systems.

4. (Simulation budget allocation) Another principle of ordinal optimization approaches is the use of different simulation lengths for different candidate solutions. The idea is to discard solutions which can hardly be optimal ones whenever we are confident enough. Figure 1 is a typical simulation time distribution when using an ordinal optimization approach. The problem of confidence probability has been addressed in [3, 19] for a class of discrete event systems.

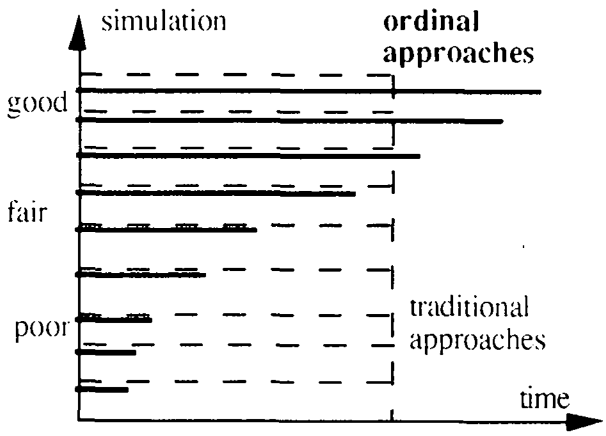

Figure 1: Simulation length profile (This figure is from [11])

Recent research ([2, 3, 14, 15, 18]) has demonstrated that impressive improvement in computation efficiency can be achieved using ordinal optimization approaches.

The purpose of this paper is to provide theoretical evidences that principles 1-3 do improve the efficiency in identifying good designs. It is an extension of a recent work ([5]) which considers the convergence of the probability that the best observed design is indeed a "good" design. However, as we point out in

Section 6, the proofs of two critical results on the exponential convergence rate of regenerative processes are inexact.

In this paper, we consider the following fundamental indicator in ordinal optimization: the probability that at least r of the observed top-k designs are the actual top-m designs (i.e. satisfactory designs). In the following, this probability is called confidence probability. It is meaningful in the perspective of dynamic simulation budget allocation. Our main contributions are the following ones :

1. Monotonicity properties of the confidence probability with respect to r, k and m are established. They show that goal relaxation improves the computation efficiency. These properties can be used in the design of ordinal optimization approaches.

2. The association (i.e. a kind of positive correlation) of different systems in simulation improves the convergence rate of the probability that the observed best design is the actual best one.

3. Informal arguments of the exponential convergence rate of the confidence probability in general case.

4. Proofs of the exponential convergence rate the confidence probability for regenerative simulation case and for empirical means of i.i.d. random variables under less restrictive conditions than the ones stated in [5].

The remainder of the paper is organized as follows. Section 2 presents the formulation of the index dynamics under consideration. Section 3 examines the monotonicity and ergodicity properties of the index dynamics. Section 4 provides a loose bound of the convergence rate, informal arguments for the exponential convergence rate, and the impact of the associated systems. Section 5 examines the convergence rate of empirical means of i.i.d. random variables and Section 6 examines that of regenerative processes.

2. Index dynamics and confidence probability

Throughout the paper, we consider the performance measure $J(\theta)$ of a discrete event system, where $\theta \in \Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$ is the design parameter of interest. We consider the following optimization problem:

$$\operatorname*{Max}_{\theta \in \Theta} J(\theta)$$

Without loss of generality, we assume that $J(\theta_1) > J(\theta_2) > \ldots > J(\theta_N)$. The term "design" will also be used to indicate a discrete event system.

The performance measures $J(\theta)$ of the various systems are unknown to the system designer and are evaluated by simulation. Let $L(\theta, \omega, t)$, $\forall t \geq 0$ denote the sample performance measure of the system θ evaluated at time t. Let $J(\theta, t) = E_\omega[L(\theta, \omega, t)]$, $\forall t \geq 0$ denote the mean value of $L(\theta, \omega, t)$. It is worth noticing that the sample performance measures $L(\theta, \omega, t)$, for all $\theta \in \Theta$ need not to be mutually independent. Furthermore, throughout the paper, for notation simplicity, we also use L_j or $L(\theta_j, t)$ to denote $L(\theta_j, \omega, t)$ when no confusion is possible.

We assume that :

(A) The discrete event systems are ergodic, i.e.

$$\lim_{t \to \infty} L(\theta, \omega, t) = J(\theta), \text{ a.s.} \quad \text{and} \quad \lim_{t \to \infty} J(\theta, t) = J(\theta)$$

At any time t of the simulation, the systems are ranked according to their sample performance measures. Let $\pi_{it} \in \Theta$ be the system ranked i at t, i.e.,

$$L(\pi_{1t}, \omega, t) \geq L(\pi_{2t}, \omega, t) \geq \ldots \geq L(\pi_{Nt}, \omega, t)$$

A fundamental issue is the consistency between the estimated ranking and the actual ranking. More precisely, we are interested in whether the observed top-k designs (i.e. estimated good designs) contain at least r of the actual top-m designs (i.e. real good designs). This is characterized by the following index dynamics:

$$I_t(r,k,m) = \begin{cases} 1, & \operatorname{card}\left(\{\theta_i : 1 \leq i \leq m\} \cap \{\pi_{it} : 1 \leq i \leq k\}\right) \geq r; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Particular cases of interest include : $I_t(r,k,k)$, $I_t(r,r,m)$, $I_t(r,k,r)$, $I_t(1,k,m)$, $I_t(r,r,r)$, $I_t(1,1,m)$. In particular, $I_t(1,1,m)$, which indicates whether the observed best design is one of the good designs, is considered in [5].

The probability measures of the index dynamics, i.e. $P[I_t(r,k,m) = 1]$ and $P[I_t(r,k,m) = 0]$, are called confidence probabilities in this paper. Our main goal is to investigate their dynamic behaviours in time.

3. Basic properties of Index dynamics and confidence probabilities

3.1. Monotonicity

Property 3.1. The index dynamics $I_t(r,k,m)$ is non-decreasing in k and m and non-increasing in r and N.

Proof. a) $I_t(r,k,m) \geq I_t(r+1,k,m)$. If the observed top-k designs contain at least r+1 of the m true best designs, i.e. $I_t(r+1,k,m) = 1$, then the observed top-k designs contain at least r of the m true best designs, i.e. $I_t(r,k,m) = 1$.

b) $I_t(r,k,m) \leq I_t(r,k+1,m)$. If $I_t(r,k,m) = 1$, then the observed top-(k+1) designs contain at least r of the m true best designs, i.e. $I_t(r,k+1,m) = 1$.

c) $I_t(r,k,m) \leq I_t(r,k,m+1)$. If $I_t(r,k,m) = 1$, then the observed top-k designs contain at least r of the m+1 true best designs, i.e. $I_t(r,k,m+1) = 1$.

d) $I_t^N(r,k,m) \geq I_t^{N+1}(r,k,m)$. Let us distinguish two cases. Case 1 : θ_{N+1} is one of the observed top-k designs. Since θ_{N+1} is not one of the m true best designs and since it is not observed when determining $I_t^N(r,k,m)$, $I_t^N(r,k,m) \geq I_t^{N+1}(r,k,m)$. Case 2 : θ_{N+1} is not one of the observed top-k designs. In this case, the observed top-k designs are independent of the design θ_{N+1} and hence $I_t^N(r,k,m) = I_t^{N+1}(r,k,m)$. ❑

Property 3.2.

a) $I_t(r,k,m) \leq I_t(r-1,k-1,m)$

b) $I_t(r,k,m) \leq I_t(r-1,k,m-1)$

c) $I_t(r,k,m) \geq I_t(r,k-1,m-1)$

Proof. If $I_t(r,k,m) = 1$, it holds that the observed top-(k-1) designs contain at least r-1 of the m true best designs, i.e. $I_t(r-1,k-1,m) = 1$, and that the observed top-k designs contain at least r-1 of the m-1 true best designs, i.e. $I_t(r-1,k,m-1) = 1$. Hence, a) and b) hold. Finally, claim c) follows directly from Property 3.1. ❑

Corollary 3.1.

$$I_t(r,k,m) \geq I_t(r,r,r)$$

Remark 3.1. Properties 3.1. and 3.2 and Corollary 3.1 still hold if $I_t(\bullet,\bullet,\bullet)$ is replaced by $P[I_t(\bullet,\bullet,\bullet) = 1]$. As a result, the confidence probability $P[I_t(r,k,m) = 1]$ can be improved by loose design requirement (i.e. small r and large m) and large computation budget (i.e. large k as it indicates the number of systems to be further evaluated). It is reduced by the number of candidate designs (i.e. N). As it can be expected, the confidence probability reaches its minimum at $r = k = m$, i.e. when only the observed top-r designs are kept and the goal is to find the actual top-r designs. That is,

Property 3.3.

a) $P[I_t(r,k,m) = 1] \geq 1 - P[I_t(r,r,r) = 0]$

b) $P[I_t(r,k,m) = 0] \leq P[I_t(r,r,r) = 0]$

Remark 3.2. Since the index dynamics considered in [5] is $I_t(1,1,m)$, the monotonicity of the confidence probabilities with respect to m and N follows from Property 3.1. The independence assumption made in [5] is not necessary.

3.2. Ergodicity

Property 3.4. Under assumption (A),

$$\lim_{t\to\infty} P\left[I_t(r,k,m) = 1\right] = 1 \quad \text{and} \quad \lim_{t\to\infty} P\left[I_t(r,k,m) = 0\right] = 0$$

This property is a direct consequence of assumption (A).

Let us notice that the confidence probabilities can be written as follows:

$$P[I_t(r,k,m) = 1] = P[a_{rm}(\omega, t) \geq b_{rk}(\omega, t)] \tag{2}$$

$$P[I_t(r,k,m) = 0] = P[a_{rm}(\omega, t) < b_{rk}(\omega, t)] \tag{3}$$

where $a_{rm}(\omega, t)$ is the sample performance measure of the observed r-th best design among $(\theta_1, \theta_2, \ldots, \theta_m)$ and $b_{rk}(\omega, t)$ is the sample performance measure of the observed (k-r+1)-th best design among $(\theta_{m+1}, \theta_{m+2}, \ldots, \theta_N)$.

Property 3.5. Under assumption (A), with probability 1, $a_{rm}(\omega, t) \to J(\theta_r)$ and $b_{rk}(\omega, t) \to J(\theta_{m+k+1})$ as $t \to \infty$.

Proof. We only consider $a_{rm}(\omega, t)$. The proof for $b_{rk}(\omega, t)$ is similar. Let Ξ be the set of all possible sample paths ω. Let $\Omega(\theta)$ be the set of sample paths ω such that $L(\theta, \omega, t)$ does not converge to $J(\theta)$ as $t \to \infty$. Clearly, for any sample path $\omega \in \Xi - \bigcup_{i=1}^{N} \Omega(\theta_i)$, $a_{rm}(\omega, t) \to J(\theta_r)$. As a result,

$$P\left[a_{rm}(\omega,t) \to J(\theta_r)\right] \geq P\left[\Xi - \bigcup_{i=1}^{N} \Omega(\theta_i)\right] \geq 1 - \sum_{i=1}^{N} P\left[\Omega(\theta_i)\right]$$

According to assumption (A), $P[\Omega(\theta)] = 0$ for all θ. Thus,

$$P\left[a_{rm}(\omega,t) \to J(\theta_r)\right] \geq 1$$

which completes the proof. ❑

Property 3.6. Under assumption (A), $E[a_{rm}(\omega, t)] \to J(\theta_r)$ and $E[b_{rk}(\omega, t)] \to J(\theta_{m+k+1})$ as $t \to \infty$.

The proof of this property is based on the notion of uniform integrability. A family of random variables $\{Y_h;\ h \in H\}$, indexed by an arbitrary set H, is uniformly integrable if

$$\sup_h E\left[|Y_h| . 1\{|Y_h| > x\}\right] \to 0 \quad \text{as } x \to \infty$$

It is known (see [8]) that uniform integrability is a necessary and sufficient condition for the interchange of limit and expectation, in the following sense: If $E[|Y_n|] < \infty$, n = 1, 2, ..., and $Y_n \to Y$ in probability, then $\lim_{n\to\infty} E[Y_n] = E[Y]$ if and only if $\{Y_n: n = 1, 2, \ldots\}$ is uniformly integrable.

Proof. Let us consider $a_{rm}(\omega, t)$. The proof for $b_{rk}(\omega, t)$ is similar. Let us notice that $a_{rm}(\omega, t) \to J(\theta_r)$ in probability according to the property (p). Furthermore,

$$E\left[\left|a_{rm}(\omega,t)\right|\right] \le E\left[\max_{1\le i\le m}\left|L(\theta_i,\omega,t)\right|\right] \le \sum_{i=1}^{m} E\left[\left|L(\theta_i,\omega,t)\right|\right] < \infty, \qquad \forall t$$

Therefore, we only need to show that $\{a_{rm}(\omega, t): t \ge 0\}$ is uniformly integrable. From the definition of $a_{rm}(\omega, t)$,

$$\begin{aligned}
&\sup_t E\left[\left|a_{rm}(\omega,t)\right| . 1\left\{\left|a_{rm}(\omega,t)\right| > x\right\}\right] \\
&\le \sup_t E\left[\max_{1\le i\le m}\left|L(\theta_i,\omega,t)\right| . 1\left\{\max_{1\le i\le m}\left|L(\theta_i,\omega,t)\right| > x\right\}\right] \\
&\le \sup_t E\left[\sum_{i=1}^{m}\left|L(\theta_i,\omega,t)\right| . 1\left\{\left|L(\theta_i,\omega,t)\right| > x\right\}\right] \\
&\le \sum_{i=1}^{m} \sup_t E\left[\left|L(\theta_i,\omega,t)\right| . 1\left\{\left|L(\theta_i,\omega,t)\right| > x\right\}\right]
\end{aligned}$$

From assumption (A), $\{L(\theta, \omega, t): t \ge 0\}$ is uniformly integrable for all θ. This leads to:

$$\sup_t E\left[\left|L(\theta,\omega,t)\right| . 1\left\{\left|L(\theta,\omega,t)\right| > x\right\}\right] \to 0 \quad \text{as } x \to \infty$$

The combination of the last two relations implies that $\{a_{rm}(\omega, t): t \ge 0\}$ is uniformly integrable. ❑

Using similar arguments, it can be shown that the estimated ranking π_{it} converges the exact ranking.

Property 3.7. Under assumption (A), $\pi_{it} \to \theta_i$ w.p.1, $L(\pi_{it}, \omega, t) \to J(\theta_i)$ w.p.1 and $E[L(\pi_{it}, \omega, t)] \to J(\theta_i)$ as $t \to \infty$.

4. Convergence rate of the general case

4.1. A loose bound

Lemma 4.1. For any real number V,

$$\text{a)}\quad P\left[I_t(r,k,m)=0\right] \le \sum_{j=1}^{r} P\left[L(\theta_j,\omega,t) \le V\right] + \sum_{j=r+1}^{N} P\left[L(\theta_j,\omega,t) \ge V\right]$$

$$\text{b)}\quad P\left[I_t(r,k,m)=1\right] \ge 1-\left(\sum_{j=1}^{r} P\left[L(\theta_j,\omega,t) \le V\right] + \sum_{j=r+1}^{N} P\left[L(\theta_j,\omega,t) \ge V\right]\right)$$

Proof. Since $P[I_t(r,k,m) = 1] = 1 - P[I_t(r,k,m) = 0]$, we only need to prove a). From Property 3.3,

$$P\left[I_t(r,k,m)=0\right] \le P\left[I_t(r,r,r)=0\right] = P\left[\operatorname*{Min}_{1\le j\le r} L_j < \operatorname*{Max}_{r+1\le j\le N} L_j\right]$$

For any real V, it holds that:

$$\begin{aligned}
P\left[\operatorname*{Min}_{1\le j\le r} L_j < \operatorname*{Max}_{r+1\le j\le N} L_j\right] &= 1-P\left[\operatorname*{Min}_{1\le j\le r} L_j \ge \operatorname*{Max}_{r+1\le j\le N} L_j\right] \\
&\le 1-P\left[\left\{\operatorname*{Min}_{1\le j\le r} L_j \ge V\right\} \cap \left\{\operatorname*{Max}_{r+1\le j\le N} L_j \le V\right\}\right] \\
&\le P\left[\left\{\operatorname*{Min}_{1\le j\le r} L_j \le V\right\} \cup \left\{\operatorname*{Max}_{r+1\le j\le N} L_j \ge V\right\}\right] \\
&\le \sum_{j=1}^{r} P\left[L(\theta_j,\omega,t) \le V\right] + \sum_{j=r+1}^{N} P\left[L(\theta_j,\omega,t) \ge V\right] .
\end{aligned}$$

which completes the proof. ❑

Property 4.1. Suppose that the variances of the sample performance measures exist. Under assumption (A), there exists a finite t* such that for all $t \ge t^*$,

$$\text{a)}\quad P\left[I_t(r,k,m)=1\right] \ge 1-\frac{1}{\Delta^2}\sum_{j=1}^{N} \operatorname{Var}(L_j)$$

$$\text{b)}\quad P\left[I_t(r,k,m)=0\right] \le \frac{1}{\Delta^2}\sum_{j=1}^{N} \operatorname{Var}(L_j)$$

where $\Delta = (J(\theta_r) - J(\theta_{r+1}))/4$.

Proof. Since $P[I_t(r,k,m) = 1] = 1 - P[I_t(r,k,m) = 0]$, we only need to prove b). From lemma 4.1, we have:

$$P\left[I_t(r,k,m)=0\right] \le \sum_{j=1}^{r} P\left[L\left(\theta_j,\omega,t\right) \le V\right] + \sum_{j=r+1}^{N} P\left[L\left(\theta_j,\omega,t\right) \ge V\right]$$

Let us choose $V = (J(\theta_r) + J(\theta_{r+1}))/2$. From assumption (A), there exists a finite t^* such that, for all $t \ge t^*$, $J(\theta_j,t) > J(\theta_j) - \Delta > V$ for all $j \le r$ and $J(\theta_j,t) < J(\theta_2) + \Delta < V$ for all $j \ge r+1$. As a result,

$$\sum_{j=1}^{r} P\left[L\left(\theta_j,\omega,t\right) \le V\right] + \sum_{j=r+1}^{N} P\left[L\left(\theta_j,\omega,t\right) \ge V\right] \le \sum_{j=1}^{N} P\left[\left|L_j - J\left(\theta_j,t\right)\right| \ge \Delta\right]$$

which implies:

$$P\left[I_t(r,k,m)=0\right] \le \sum_{j=1}^{N} P\left[\left|L_j - J\left(\theta_j,t\right)\right| \ge \Delta\right] \tag{4}$$

Combining this relation with Chebyshev's inequality completes the proof. □

The implication of this property is that the confidence probabilities converge at least as fast as the variances of the performance measures. Typically, the convergence rate of the variances is $O(1/t)$. If this is true, then $P[I_t(r,k,m) = 1] = 1 - O(1/t)$ and $P[I_t(r,k,m) = 0] = O(1/t)$.

Let us notice that as shown in [5], this lower bound cannot be improved if only the first two moments are known.

4.2. Informal arguments of the exponential convergence rate

Let us further examine relation (4) which can be rewritten as follows:

$$P\left[I_t(r,k,m)=0\right] \le \sum_{j=1}^{N} P\left[\left|L\left(\theta_j,\omega,t\right) - J\left(\theta_j,t\right)\right| \ge \Delta\right] \tag{4'}$$

As a result, the confidence probabilities converge at exponential rate if the sample performance measures converge at exponential rate or if they satisfy the large deviation principle. This has been proven true for many ergodic stochastic processes. Hereunder, we give some informal arguments.

For ergodic stochastic process, it is generally true that the sample performance measures are asymptotically normally distributed. It is then reasonable to assume that $L(\theta, \omega, t)$ is asymptotically normally distributed, i.e.

$$P[\,|L(\theta,\omega,t) - J(\theta,t)| \geq \Delta] \approx P[\,|v(\theta,t) - \mu_t| \geq \Delta], \text{ for t large enough}$$

where $v(\theta, t)$ is normally distributed with mean $\mu_t = J(\theta,t)$ and variance $(\sigma_t)^2 = \mathrm{Var}(L(\theta, \omega, t))$. According to a result in large deviation theory (see [6]),

$$P[\,|v(\theta,t) - \mu_t| \geq \Delta] \leq 2\mathbf{exp}(-\Lambda^*(\Delta))$$

where $\Lambda^*(\Delta) = \Delta^2/(\sigma_t)^2$. As a result, for t large enough,

$$P[\,|L(\theta,\omega,t) - J(\theta,t)| \geq \Delta] \leq 2\mathbf{exp}(-\Lambda^*(\Delta))$$

which together with relation (4') implies that :

$$P[I_t(r,k,m) = 0] \approx O(\mathbf{exp}(-s(t)))$$

where $s(t) = \Delta^2/\mathrm{Max}(\mathrm{Var}(L(\theta, \omega, t)) : \theta \in \Theta)$.

If the variances of the performance measures converge to 0 at rate $O(1/t)$, then $P[I_t(r,k,m) = 0]$ converges to 0 at rate $\mathbf{exp}(-O(t))$.

4.3. Determination of the exponential convergence rate

Lemma 4.2. For any random variable e_t such that $e_t \to \mu$ a.s., $E[e_t] \to \mu$ and $\mu < 0$. There exists $t_0 \geq 0$ such that :

$$P[e_t \geq 0] \leq \exp(-t\Lambda^*(t)), \text{ for all } t \geq t_0$$

where

$$\Lambda^*(t) = \sup_{\lambda}\left\{-\frac{1}{t}\log M_t(\lambda)\right\} \quad \text{and} \quad M_t(\lambda) = E\left[e^{\lambda e_t}\right]$$

Proof. First, for all $\lambda \geq 0$,

$$P[e_t \geq 0] = E[1\{e_t \geq 0\}] \leq E\left[e^{\lambda e_t}\right] = M_t(\lambda)$$

which implies that:

$$P[e_t \geq 0] \leq \exp\left(-\sup_{\lambda \geq 0}\left\{-\frac{1}{t}\log M_t(\lambda)\right\}\right)$$

Since $E[e_t] \to \mu$ and $\mu < 0$, there exists $t_0 \geq 0$ such that :

$$E[e_t] \leq 0, \text{ for all } t \geq t_0$$

From the properties of the Fenchel-Legendre transform,

$$P[e_t \geq 0] \leq \exp\left(-t\sup_{\lambda}\left\{-\frac{1}{t}\log M_t(\lambda)\right\}\right) = \exp(-t\Lambda^*(t))$$

□

We use relations (2) and (3) to determine the convergence rates of the confidence probabilities. Rewrite equation (3) as follows:

$$P[I_t(r,k,m) = 0] = P[a_{rm}(\omega, t) < b_{rk}(\omega, t)]$$

where $a_{rm}(\omega, t)$ is the sample performance measure of the observed r-th best design among $(\theta_1, \theta_2, \ldots, \theta_m)$ and $b_{rk}(\omega, t)$ is the sample performance measure of the observed (k-r+1)-th best design among $(\theta_{m+1}, \theta_{m+2}, \ldots, \theta_N)$. Let

$$e_t(r,k,m) = b_{rk}(\omega, t) - a_{rm}(\omega, t)$$

According to properties 3.5-3.6, the conditions of lemma 4.2 hold for $e_t(r,k,m)$. As a result, there exists $t_0 \geq 0$ such that :

$$P[I_t(r,k,m) = 0] \leq \exp(-tH_t(r,k,m)), \text{ for all } t \geq t_0 \tag{5}$$

where

$$H_t(r,k,m) = \sup_{\lambda}\left\{-\frac{1}{t}\log E\left[\exp\left(\lambda\left(b_{rk}(\omega,t) - a_{rm}(\omega,t)\right)\right)\right]\right\} \tag{6}$$

The exponential convergence rates of the confidence probabilities satisfy the following relation:

$$\lim_{t\to\infty}\frac{1}{t}\log P[I_t(r,k,m) = 0] \leq -\lim_{t\to\infty} H_t(r,k,m) = -H_\infty(r,k,m)$$

From the numerous results in the theory of large deviation, it is commonly believed that the above inequality relation is actually an equality relation and that the right-hand-side term of this relation is the exponential convergence rate of the related confidence probability.

4.4. Association and convergence rate

Lemma 4.3. If $\{L(\theta, \omega, t)) : \theta \in \Theta\}$ is a set of associated random variables, then the random variables $b_{rk}(\omega, t)$ and $a_{rm}(\omega, t)$ are associated.

Proof. Obvious as $a_{rm}(\omega, t)$ and $b_{rk}(\omega, t)$ are monotone non-decreasing functions of $L(\theta, \omega, t))$ for all $\theta \in \Theta$. ❑

Let $\{\overline{L}(\theta,\omega,t) : \theta \in \Theta\}$ be an independent version of $\{L(\theta, \omega, t) : \theta \in \Theta\}$ in the sense of [1]. As a result,

$$\mathrm{Var}(L(\theta, \omega, t) - L(\theta', \omega, t)) \leq \mathrm{Var}(\overline{L}(\theta,\omega,t) - \overline{L}(\theta',\omega,t))$$

However, as noticed in [5, 9], small variance does not imply high probability of correct selection, i.e. higher $P[I_t(r,k,m) = 1]$. Nevertheless, as will be shown in the following, when we check whether the observed best is the actual best, the association does lead to large rate of convergence which means that the confidence probabilities converge faster in the simulation of associated systems than in that of independent systems.

Let $A_{rm}(\omega, t)$ is the sample performance measure $\overline{L}(\theta,\omega,t)$ of the observed r-th best design among $(\theta_1, \theta_2, \ldots, \theta_m)$ and $B_{rk}(\omega, t)$ is the sample performance measure $\overline{L}(\theta,\omega,t)$ of the observed (k-r+1)-th best design among $(\theta_{m+1}, \theta_{m+2}, \ldots, \theta_N)$. From their definition, $A_{rm}(\omega,t)$ and $B_{rk}(\omega,t)$ are mutually independent. Clearly, if r = m =k = 1,

$$A_{rm(\omega,t)} = \overline{L}(\theta_1,\omega,t) \quad \text{and} \quad B_{rk}(\omega,t) = \underset{2\leq j\leq N}{\mathrm{Max}}\, \overline{L}(\theta_j,\omega,t)$$

If $\{L(\theta, \omega, t)) : \theta \in \Theta\}$ is a set of associated random variables, according to [1]:

$$a_{rm(\omega,t)} =_{st} A_{rm(\omega,t)} \quad \text{and} \quad b_{rm(\omega,t)} \leq_{st} B_{rm(\omega,t)} \tag{7}$$

Property 4.2. If $\{L(\theta, \omega, t)) : \theta \in \Theta\}$ is a set of associated random variables, then for t large enough,

$$H_t(1,1,1) \geq \overline{H}_t(1,1,1)$$

where

$$\overline{H}_t(r,k,m) = \sup_{\lambda}\left\{-\frac{1}{t}\log E\left[\exp\left(\lambda(B_{rk}(\omega,t) - A_{rm}(\omega,t))\right)\right]\right\}$$

Proof. For all value of λ, one of the two functions $x \to \exp(\lambda x)$ and $y \to \exp(-\lambda y)$ is increasing and the other is decreasing. Since $a_{rm}(\omega, t)$ and $b_{rk}(\omega, t)$ are associated,

$$E\left[\exp\left(\lambda(b_{rk}(\omega,t) - a_{rm}(\omega,t))\right)\right] \le E\left[\exp(\lambda b_{rk}(\omega,t))\right]E\left[\exp(-\lambda a_{rm}(\omega,t))\right]$$

Since $A_{rm}(\omega,t)$ and $B_{rk}(\omega,t)$ are mutually independent,

$$E\left[\exp\left(\lambda(B_{rk}(\omega,t) - A_{rm}(\omega,t))\right)\right] = E\left[\exp(\lambda B_{rk}(\omega,t))\right]E\left[\exp(-\lambda A_{rm}(\omega,t))\right]$$

When r=k=m=1, combining the last two relations and relation (7), we obtain

$$E\left[\exp\left(\lambda(b_{rk}(\omega,t) - a_{rm}(\omega,t))\right)\right] \le E\left[\exp\left(\lambda(B_{rk}(\omega,t) - A_{rm}(\omega,t))\right)\right], \quad \forall\lambda \ge 0$$

From property 3.6, $E[a_{rm}(\omega, t)] \to J(\theta_r)$, $E[b_{rk}(\omega, t)] \to J(\theta_{m+k+1})$, $E[A_{rm}(\omega, t)] \to J(\theta_r)$ and $E[B_{rk}(\omega, t)] \to J(\theta_{m+k+1})$ as $t \to \infty$. As a result, there exists a finite t^* such that, for all $t \ge t^*$, $E[b_{rm}(\omega,t) - a_{rm}(\omega,t)] \le 0$ and $E[B_{rm}(\omega,t) - A_{rm}(\omega,t)] \le 0$. According to Lemma 2.2.5 in [6], it holds that:

$$H_t(r,k,m) = \sup_{\lambda \ge 0}\left\{-\frac{1}{t}\log E\left[\exp\left(\lambda(b_{rk}(\omega,t) - a_{rm}(\omega,t))\right)\right]\right\}, \quad \forall t \ge t^*$$

$$\overline{H}_t(r,k,m) = \sup_{\lambda \ge 0}\left\{-\frac{1}{t}\log E\left[\exp\left(\lambda(B_{rk}(\omega,t) - A_{rm}(\omega,t))\right)\right]\right\}, \quad \forall t \ge t^*$$

which completes the proof. ❑

Remark 4.1. Similar property was established in [9] for the case of two Markov chains. The proof of property 4.2 above is similar to that of lemma 5.2 in [9].

Remark 4.2. Numerous numerical experiences show that property also holds for general case. This is an important open problem.

4.5. Some large deviation results

This subsection establishes some preliminary results that will be extensively used. For this purpose, let us consider a sequence of i.i.d. random variables X_1, ..., X_n, ... Let us consider the following assumption:

(B) The moment generating function $M(\lambda) = E[\mathbf{exp}(\lambda X_1)]$ exists in a neighbourhood $\Omega = (-\varpi, \varpi)$ of $\lambda = 0$ for some $\varpi > 0$.

Let $S_n = \sum_{i=1}^{n} X_i$, $\Lambda(\lambda) = \log M(\lambda)$ and $\Lambda^*(x)$ be the Fenchel-Lengendre transform of $\Lambda(\lambda)$, i.e.

$$\Lambda^*(x) = \sup_{\lambda}\{\lambda\ x - \Lambda(\lambda)\}$$

The first quantity of interest is the probability that the empirical mean, i.e. S_n/n, is greater (resp. smaller) than a given finite constant. This quantity converges to zero at exponential rate according to the following lemma.

Lemma 4.4.
If (B) holds, then

$$P[S_n/n \geq x] \leq \exp(-n\Lambda^*(x)),\ \forall x \geq E[X_1]$$

and

$$P[S_n/n \leq x] \leq \exp(-n\Lambda^*(x)),\ \forall x \leq E[X_1]$$

Furthermore, $\Lambda^*(x) > 0$, $\forall x \neq E[X_1]$.

Proof : Let us notice that assumption (B) implies that $\Lambda(\lambda)$ exists in Ω. According to Lemma 2.2.5 in [6], $\Lambda^*(x)$ is a non negative convex function with $\Lambda^*(E[X_1]) = 0$ and

$$\Lambda^*(x) = \begin{cases} \sup_{\lambda \geq 0}\{\lambda\ x - \Lambda(\lambda)\}, & \text{if } x \geq E[X_1] \\ \sup_{\lambda \leq 0}\{\lambda\ x - \Lambda(\lambda)\}, & \text{if } x \leq E[X_1] \end{cases}$$

Furthermore, $\Lambda(\lambda)$ is a convex function. It is differentiable in Ω and $\Lambda'(0) = E[X_1]$. For any $x > E[X_1]$,

$$\begin{aligned} \Lambda^*(x) &\geq \sup_{0 \leq \lambda \leq \varpi}\{\lambda\ x - \Lambda(\lambda)\} \\ &= \sup_{0 \leq \lambda \leq \varpi}\{\lambda\ x - (\Lambda(0) + \Lambda'(0)\lambda + o(\lambda))\} \\ &= \sup_{0 \leq \lambda \leq \varpi}\{\lambda\ (x - E[X_1]) + o(\lambda)\} \\ &> 0 \end{aligned}$$

Similarly, it can be shown that $\Lambda^*(x) > 0$, $\forall x < E[X_1]$. Finally, it can be easily shown that:

$$P[S_n/n \geq x] \leq \inf_{\lambda \geq 0}\{E[\mathbf{exp}(\lambda(S_n - nx))]\} = \mathbf{exp}(-n\Lambda^*(x)),\ \forall x \leq E[X_1]$$

and

$$P[S_n/n \leq x] \leq \inf_{\lambda \leq 0} \{E[\mathbf{exp}(\lambda(S_n - nx))]\} = \mathbf{exp}(-n\Lambda^*(x)), \quad \forall x \leq E[X_1]$$

Q.E.D.

Let us notice that Cramer's theorem provides stronger results. As a matter of fact, it shows that:

$$\lim_{n\to\infty} \frac{1}{n} \log P[S_n / n \geq x] = -\Lambda^*(x), \forall x \geq E[X_1]$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log P[S_n / n \leq x] = -\Lambda^*(x), \forall x \leq E[X_1]$$

Another quantity of interest is the probability that a random variable exceeds a large number. As shown in the following lemma, it also converges to zero at exponential rate if assumption (B) holds.

Lemma 4.5.

$$P[X_1 \geq x] \leq M(\lambda) \exp(-\lambda x), \forall \lambda \geq 0$$

and

$$P[X_1 \leq -x] \leq M(\lambda) \exp(\lambda x), \forall \lambda \leq 0$$

5. Empirical means of i.i.d. random variables

In this section, we consider the following performance measure:

$$L(\theta,t) = \frac{1}{t}\sum_{i=1}^{t} X_i(\theta)$$

where $\{X_i(\theta), i \geq 1\}$ is a sequence of i.i.d. random variables with $E[X_i(\theta)] < \infty$.

Theorem 5.1. Assume that the moment generating function $M(\lambda,\theta) = E[\exp(\lambda X_1(\theta))]$ exists in a neighbourhood $\Omega = (-\varpi, \varpi)$ of $\lambda = 0$ for some $\varpi > 0$. Then there exists a positive number $s > 0$ such that :

$$P[I_t(r,k,m) = 1] = 1 - O(\exp(-st)) \text{ and } P[I_t(r,k,m) = 0] = O(\exp(-st)).$$

Proof. Since $P[I_t(r,k,m) = 1] = 1 - P[I_t(r,k,m) = 0]$, we only need to prove b). From lemma 4.1, we have:

$$P[I_t(r,k,m) = 0] \leq \sum_{j=1}^{r} P[L(\theta_j,\omega,t) \leq V] + \sum_{j=r+1}^{N} P[L(\theta_j,\omega,t) \geq V]$$

Let $V = (J(\theta_r) + J(\theta_{r+1}))/2$ and let $\Delta = (J(\theta_r) - J(\theta_{r+1}))/2$. Since $J(\theta_1) > J(\theta_2) > \ldots > J(\theta_N)$, it can be shown that:

$$P[I_t(r,k,m)=0] \le \sum_{j=1}^{r} P[L(\theta_j,t) \le J(\theta_j) - \Delta] + \sum_{j=r+1}^{N} P[L(\theta_j,t) \ge J(\theta_j) + \Delta] \quad (8)$$

Taking into account the form of the performance measures,

$$P[I_t(r,k,m)=0] \le \sum_{j=1}^{r} P\left[\frac{1}{t}\sum_{i=1}^{t} X_i(\theta_j) \le J(\theta_j) - \Delta\right] + \sum_{j=r+1}^{N} P\left[\frac{1}{t}\sum_{i=1}^{t} X_i(\theta_j) \ge J(\theta_j) + \Delta\right]$$

According to lemma 4.4,

$$P\left[\frac{1}{t}\sum_{i=1}^{t} X_i(\theta_j) \ge J(\theta_j) + \Delta\right] \le \exp\left(-t\Lambda_j^*(J(\theta_j)+\Delta)\right)$$

$$P\left[\frac{1}{t}\sum_{i=1}^{t} X_i(\theta_1) \le J(\theta_1) - \Delta\right] \le \exp\left(-t\Lambda_1^*(J(\theta_1)-\Delta)\right)$$

where

$$\Lambda_k^*(x) = \sup_{\lambda}\left\{\lambda x - \log E[\exp(\lambda X_1(\theta_k))]\right\} > 0, \qquad \forall x \ne E[X_1(\theta_k)]$$

Combining the above results,

$$P[I_t(r,k,m) = 0] = O(\exp(-st))$$

where $s = \mathrm{Min}\left\{\Lambda_1^*(J(\theta_1)-\Delta), \ldots, \Lambda_r^*(J(\theta_r)-\Delta), \Lambda_{r+1}^*(J(\theta_{r+1})+\Delta), \ldots, \Lambda_N^*(J(\theta_N)+\Delta)\right\}$ is a positive number by virtue of lemma 4.4. ❑

Remark 5.1. The above result can be easily extended to the case where the number t depends on θ, i.e.

$$L(\theta,\omega,t) = \frac{1}{N(\theta,t)} \sum_{i=1}^{N(\theta,t)} X_i(\theta,\omega)$$

In this case,

$$P[I_t(r,k,m) = 1] = 1 - O(\exp(-sv_t)) \text{ and } P[I_t(r,k,m) = 0] = O(\exp(-sv_t)).$$

where

$$v_t = \mathrm{Min}\{N(\theta_1,t), \ldots, N(\theta_N,t)\}.$$

Remark 5.2. The different sequences of i.i.d. random variables need not to be mutually independent.

Remark 5.3. The condition of the theorem 5.1 is less restrictive than the conditions needed in [5].

6. Regenerative simulation

The following notation will be used in this section :

$\{S_i(\theta) : i \geq 0\}$: regeneration times. $S_0(\theta)$ is the initial delay;
$\{Y_i(\theta) : i \geq 0\}$: regeneration cycles. Of course, $S_i(\theta) = \Sigma_{j \leq i} Y_i(\theta)$

We consider three performance measures :

Classical estimator :

$$L_1(\theta) = \frac{\sum_{i=1}^{K(\theta)} X_i(\theta)}{\sum_{i=1}^{K(\theta)} Y_i(\theta)}$$

Heidelberger and Meketon's estimator ([9]) :

$$L_2(\theta,t) = \frac{\sum_{i=1}^{K(\theta,t)+1} X_i(\theta)}{\sum_{i=1}^{K(\theta,t)+1} Y_i(\theta)}$$

Time average :

$$L(\theta,t) = \frac{1}{t}\int_{s=0}^{t} l_t(\theta)ds$$

where $K(\theta)$ is given number, $K(\theta,t)$ is a number of regeneration cycles completed by time t and

$$X_i(\theta) = \int_{s=S_{i-1}(\theta)}^{S_i(\theta)} l_t(\theta)ds$$

We assume that the regeneration process has i.i.d. cycles which implies that $\{(Y_i(\theta), X_i(\theta)) : i \geq 1\}$ is a sequence of i.i.d. random variables. We further assume that $|l_t(\theta)| \leq B$.

Theorem 6.1. Assume that the classical estimators $L_1(\theta)$ are used. Assume that the moment generating functions $M(\lambda,\theta) = E[\exp(\lambda Y_1(\theta,\omega))]$ exist in a neighbourhood $\Omega = (-\varpi, \varpi)$ of $\lambda = 0$ for some $\varpi > 0$. Then there exists a positive number $s > 0$ such that :

$$P[I(r,k,m) = 1] = 1 - O(\exp(- sK^*)) \text{ and } P[I(r,k,m) = 0] = O(\exp(- sK^*))$$

where $K^* = \mathrm{Min}\{K(\theta_i) : 1 \le i \le N\}$.

Proof. Since $P[I_t(r,k,m) = 1] = 1 - P[I_t(r,k,m) = 0]$, we only need to prove b). We notice that relation (8) holds whatever the performance measures. Hence:

$$P[I_t(r,k,m)=0] \le \sum_{j=1}^{r} P[L(\theta_j,t) \le J(\theta_j)-\Delta] + \sum_{j=r+1}^{N} P[L(\theta_j,t) \ge J(\theta_j)+\Delta]$$

where $\Delta = (J(\theta_r) - J(\theta_{r+1}))/2$.

Clearly,

$$P[L(\theta,t) \ge J(\theta)+\Delta] = P\left[\frac{\sum_{i=1}^{K(\theta)} X_i(\theta)}{\sum_{i=1}^{K(\theta)} Y_i(\theta)} \ge J(\theta)+\Delta\right] = P\left[\sum_{i=1}^{K(\theta)} Z_i(\theta) \ge 0\right]$$

where $Z_i(\theta) = X_i(\theta) - (J(\theta) + \Delta)\, Y_i(\theta)$. $\{Z_i(\theta) : i \ge 0\}$ is a sequence of i.i.d. random variables. Furthermore, since $|l_t(\theta)| \le B$,

$$|Z_i(\theta)| = |X_i(\theta) - (J(\theta) + \Delta)\, Y_i(\theta)| \le 2BY_i(\theta)$$

As a result, $E[\exp(\lambda Z_i(\theta)]$ exists in $\Omega' = (-\varpi/2B, \varpi/2B)$. Since

$$E[Z_i(\theta)] = \Delta E[Y_1(\theta)] < 0,$$

according to lemma 4.4,

$$P[L(\theta,t) \ge J(\theta)+\Delta] = P\left[\sum_{i=1}^{K(\theta)} Z_i(\theta) \ge 0\right] \le \exp\left(-K(\theta)\Lambda_\theta^*\right)$$

where

$$\Lambda_\theta^* = \sup_\lambda\left\{-\log E\left[\exp(\lambda Z_1(\theta))\right]\right\} > 0$$

Similarly, it can be shown that :

$$P[L(\theta,t) \le J(\theta)-\Delta] \le \exp\left(-K(\theta)h_\theta^*\right)$$

where

$$h_\theta^* = \sup_\lambda\left\{-\log E\left[\exp\left(\lambda\left(X_1(\theta)-(J(\theta)-\Delta)Y_1(\theta)\right)\right)\right]\right\} > 0$$

Combining the above results, we obtain

$$P[I_t(r,k,m) = 0] = O(\exp(-sK^*))$$

where

$$s = \operatorname{Min}\left\{h^*_{\theta_1},\dots,h^*_{\theta_r},\Lambda^*_{\theta_{r+1}},\dots,\Lambda^*_{\theta_N}\right\}. \qquad \square$$

Theorem 6.2. Assume that the classical estimators $L_2(\theta, t)$ are used. Assume that the moment generating functions $m(\lambda,\theta) = E[\exp(\lambda Y_0(\theta,\omega))]$ and $M(\lambda,\theta) = E[\exp(\lambda Y_1(\theta,\omega))]$ exist in a neighbourhood $\Omega = (-\varpi, \varpi)$ of $\lambda = 0$ for some $\varpi > 0$. Then there exists a positive number $s > 0$ such that :

$$P[I_t(r,k,m) = 1] = 1 - O(\exp(-st)) \quad \text{and} \quad P[I_i(r,k,m) = 0] = O(\exp(-st))$$

Proof. As in the proof of theorem 6.1, we have:

$$P[I_t(r,k,m)=0] \le \sum_{j=1}^{r} P[L(\theta_j,t) \le J(\theta_j)-\Delta] + \sum_{j=r+1}^{N} P[L(\theta_j,t) \ge J(\theta_j)+\Delta]$$

where $\Delta = (J(\theta_r) - J(\theta_{r+1}))/2$. Furthermore,

$$P[L(\theta,t) \ge J(\theta)+\Delta] = P\left[\sum_{i=1}^{K(\theta,t)+1} Z_i(\theta) \ge 0\right]$$

$$P[L(\theta,t) \le J(\theta)-\Delta] = P\left[\sum_{i=1}^{K(\theta,t)+1} W_i(\theta) \ge 0\right]$$

where $Z_i(\theta) = X_i(\theta) - (J(\theta) + \Delta)\, Y_i(\theta)$ and $W_i(\theta) = X_i(\theta) - (J(\theta) - \Delta)\, Y_i(\theta)$.

For any integer $Q > 0$, we have:

$$P\left[\Psi_{K(\theta,t)+1} \ge 0\right] = P\left[\bigcup_{n=0}^{\infty} \Psi_{n+1} \ge 0 \cap K(\theta,t) = n\right]$$

$$\le P\left[\bigcup_{n=0}^{Q-1} \Psi_{n+1} \ge 0 \cap K(\theta,t) = n\right] + P\left[\bigcup_{n=Q}^{\infty} \Psi_{n+1} \ge 0 \cap K(\theta,t) = n\right]$$

$$\le P[K(\theta,t) < Q] + P\left[\bigcup_{n=Q}^{\infty} \Psi_{n+1} \ge 0\right]$$

where $\Psi_k = \sum_{i=1}^{k} Z_i(\theta)$.

From the proof of theorem 6.1,

$$P\left[\bigcup_{n=Q}^{\infty} \Psi_{n+1} \ge 0\right] = \sum_{n=Q+1}^{\infty} P[\Psi_n \ge 0] \le \sum_{n=Q+1}^{\infty} \exp(-n\gamma(\theta)) = \frac{\exp(-\gamma(\theta)(Q+1))}{1-\exp(-\gamma(\theta))}$$

where

$$\gamma(\theta) = \sup_{\lambda}\left\{-\log E\left[\exp(\lambda Z_1(\theta))\right]\right\} > 0$$

From the definition of the renewal number K(θ,t),

$$P\left[K(\theta,t) < Q\ \right] = P\left[S_Q > t\ \right] = P\left[\ Y_0 + \sum_{i=1}^{Q} Y_i > t\right]$$

which leads to:

$$P\left[K(\theta,t) < Q\ \right] \le P\left[\ Y_0 \ge \frac{t}{2}\right] + P\left[\sum_{i=1}^{Q} Y_i \ge \frac{t}{2}\right]$$

By applying lemma 4.4 to the second term on the RHS and lemma 4.5 to the first term, it can be shown that, if $QE[Y_1] < t/2$,

$$P\left[K(\theta,t) < Q\ \right] \le m(\mu,\theta)\exp(-\mu t/2) + \exp\left(-Q\Lambda^*(\theta)\right)$$

where μ is any positive real in Ω and

$$\Lambda^*(\theta) = \sup_{\lambda}\left\{\lambda\frac{t}{2Q} - \log M(\lambda,\theta)\right\} > 0$$

By choosing $Q = \left\lfloor t/\left(4E\left[Y_1\right]\right)\right\rfloor$ and by combining the above results, we have:

$$P\left[L(\theta,t) \ge J(\theta) + \Delta\right] \le P\left[K(\theta,t) < Q\ \right] + P\left[\ \bigcup_{n=Q}^{\infty}\Psi_{n+1} \ge 0\ \right] = O\left(\exp(-s(\theta)t\right)$$

where

$$s(\theta) = \mathrm{Min}\left\{\frac{\gamma(\theta)}{4E\left[Y_1\right]}, \frac{\mu}{2}, \frac{\Lambda^*(\theta)}{4E\left[Y_1\right]}\right\} > 0$$

Similarly, it can be shown that there exists a positive real a(θ) > 0 such that :

$$P\left[L(\theta,t) \le J(\theta) - \Delta\right] \le= O\left(\exp(-a(\theta)t\right)$$

which implies that:

$$P\left[I_t(r,k,m) = 0\right] \le O\left(\exp\left(-t\cdot \mathrm{Min}\left\{a(\theta_1),\ldots,a(\theta_r),s(\theta_{r+1}),\ldots,s(\theta_N)\right\}\right)\right) \qquad \square$$

Remark 6.1. The theorem 6.2 still holds if the terms K(θ,t) + 1 in Heidelberger and Meketon's estimator $L_2(\theta,t)$ are replaced by K(θ,t). The proof is straightforward.

Remark 6.2. The performance measures of the different systems can be evaluated at different point of the time, i.e. t may depend on θ. In this case, the convergence rate becomes exp(-s t*) with t* = Min{t(θ)}.

Remark 6.3. In the proof of theorem, since

$$P\left[\sum_{i=1}^{K(\theta,t)+1} Z_i(\theta) \geq 0 \right] = E\left[P\left[\sum_{i=1}^{K(\theta,t)+1} Z_i(\theta) \geq 0 \,|K(\theta,t)\right]\right],$$

it is tempting to apply theorem 6.2 to compute the convergence rate of the term P[•] on the RHS. However, it is known that, when conditioning on $K(\theta,t)$, the random variables $Z_i(\theta)$ for all $i \leq K(\theta,t) + 1$ are no longer i.i.d. Hence theorem 6.1 does not apply. We notice that similar inexact arguments were used in the proofs of theorems 4.5 and 4.6 in [5]. As a matter of fact, the following inexact relation was used in [5]:

$$E\left[\exp\left(s\sum_{i=1}^{K(\theta,t)} m_i(\theta)\right)|K(\theta,t)\right] = \prod_{i=1}^{K(\theta,t)} E\left[\exp(sm_i(\theta))\right]$$

where $m_i(\theta) = X_i(\theta) - Y_i(\theta)J(\theta)$.

Theorem 6.3. Assume that the time average estimators $L(\theta, t)$ are used. Assume that the moment generating functions $m(\lambda,\theta) = E[\exp(\lambda Y_0(\theta,\omega))]$ and $M(\lambda,\theta) = E[\exp(\lambda Y_1(\theta,\omega))]$ exist in a neighbourhood $\Omega = (-\varpi, \varpi)$ of $\lambda = 0$ for some $\varpi > 0$. Then there exists a positive number $s > 0$ such that :

$$P[I_t(r,k,m) = 1] = 1 - O(\exp(- st)) \text{ and } P[I_i(r,k,m) = 0] = O(\exp(- st))$$

Proof. First, let us notice that $L(\theta,t)$ can be rewritten as follows:

$$L(\theta,t) = \frac{1}{t}\left(X_0(\theta) + \sum_{i=1}^{K(\theta,t)} X_i(\theta) + W_t(\theta)\right) \tag{9}$$

where

$$X_0(\theta) = \int_{s=0}^{Y_0(\theta)} 1_t(\theta)ds \qquad \text{and} \qquad W_t(\theta) = \int_{s=S_{K(\theta,t)}(\theta)}^{t} 1_t(\theta)ds$$

As in the proof of theorem 6.1, we have:

$$P\left[I_t(r,k,m) = 0\right] \leq \sum_{j=1}^{r} P\left[L(\theta_j,t) \leq J(\theta_j) - \Delta\right] + \sum_{j=r+1}^{N} P\left[L(\theta_j,t) \geq J(\theta_j) + \Delta\right]$$

where $\Delta = (J(\theta_r) - J(\theta_{r+1}))/2$.

Denote :

$$P_1(\alpha) = P\left[L(\alpha,t) \leq J(\alpha) - \Delta\right]$$
$$P_2(\alpha) = P\left[L(\alpha,t) \geq J(\alpha) + \Delta\right]$$

Clearly, $P[I_i(r,k,m) = 0]$ converges to zero at exponential rate if both $P_1(\alpha)$ and $P_2(\alpha)$ do for all $\alpha \in \Theta$. In the following, only $P_2(\alpha)$ is considered and the convergence rate of $P_1(\alpha)$ can be established in a similar way.

By expressing $L(\alpha,t)$ in the form of equation (9), we obtain:

$$P_2(\alpha) = P[L(\alpha,t) \geq J(\alpha) + \Delta] = P\left[\frac{1}{t}\left(Z_0(\alpha) + \sum_{i=1}^{K(\alpha,t)} Z_i(\alpha) + V_t(\alpha)\right) \geq \Delta\right]$$

where $Z_i(\alpha) = X_i(\alpha) - J(\alpha)Y_i(\alpha)$ and $V_t(\alpha) = W_t(\alpha) - J(\alpha)(t - S_{K(\alpha,t)}(\alpha))$. It follows:

$$P_2(\alpha) \leq P\left[\frac{Z_0(\alpha)}{t} \geq \frac{\Delta}{3}\right] + P\left[\frac{\sum_{i=1}^{K(\alpha,t)} Z_i(\alpha)}{t} \geq \frac{\Delta}{3}\right] + P\left[\frac{V_t(\alpha)}{t} \geq \frac{\Delta}{3}\right].$$

Clearly, it is enough to show that each of the three terms on the RHS of the above relation converges at exponential rate.

Since $t \geq S_{K(\alpha,t)} \geq \sum_{i=1}^{K(\alpha,t)} Y_i(\alpha)$, we have :

$$P\left[\frac{1}{t}\sum_{i=1}^{K(\alpha,t)} Z_i(\alpha) \geq \frac{\Delta}{3}\right] \leq P\left[\frac{\sum_{i=1}^{K(\alpha,t)} Z_i(\alpha)}{\sum_{i=1}^{K(\alpha,t)} Y_i(\alpha)} \geq \frac{\Delta}{3}\right]$$

Following the proof of theorem 6.2, it can be shown that there exists a positive real $s_1 > 0$ such that :

$$P\left[\frac{1}{t}\sum_{i=1}^{K(\alpha,t)} Z_i(\alpha) \geq \frac{\Delta}{3}\right] \leq P\left[\frac{\sum_{i=1}^{K(\alpha,t)} Z_i(\alpha)}{\sum_{i=1}^{K(\alpha,t)} Y_i(\alpha)} \geq \frac{\Delta}{3}\right] \leq \exp(-s_1 t)$$

Since $|l_i(\alpha)| \leq B$, $|Z_0(\alpha)| \leq 2B\, Y_i(\alpha)$ which implies that $E[\exp(\lambda Z_0(\alpha))]$ exists in $(-\varpi/2B, \varpi/2B)$. From lemma 4.5, for any $0 < \mu < \varpi/2B$,

$$P\left[\frac{1}{t} Z_0(\alpha) \geq \frac{\Delta}{3}\right] \leq E\left[\exp(\mu Z_0(\alpha))\right]\exp(-\mu\Delta t / 3)$$

In order to determine the convergence rate of $P[V_t(\alpha)/t \geq \Delta/3]$, we need to prove that $E[\exp(\lambda V_t(\alpha))]$ exists for some $\lambda > 0$. Let $A_t = (t - S_{K(\alpha,t)}(\alpha))$ be the age at time t of the regenerative process. From the definition of $V_t(\alpha)$, $|V_t(\alpha)| \leq 2BA_t$ which implies that $E[\exp(\lambda V_t(\alpha))]$ exists whenever $E[\exp(|\lambda| 2BA_t)]$ exists. The moment generating function $E[\exp(\lambda A_t)]$ can be determined by using a renewal reward process in which the reward is accumulated at rate $\exp(\lambda A_t)$. Then,

$$\lim_{t\to\infty} E\left[\exp(\lambda A_t)\right] = \frac{E\left[\int_{s=0}^{Y_1(\alpha)} \exp(\lambda s)ds\right]}{E\left[Y_1(\alpha)\right]}$$

whenever both expectations on the RHS exist (see [16]). As a result,

$$\lim_{t\to\infty} E\left[\exp(\lambda A_t)\right] = \frac{M(\lambda,\alpha)-1}{\lambda E\left[Y_1(\alpha)\right]}, \quad \forall 0 < \lambda < \varpi$$

which implies that there exists a finite $\varphi(\lambda) \geq 0$ such that

$$E\left[\exp(\lambda A_t)\right] \leq 2\frac{M(\lambda,\alpha)-1}{\lambda E\left[Y_1(\alpha)\right]}, \qquad \forall 0 < \lambda < \varpi, \forall t \geq \varphi(\lambda)$$

Therefore, $E[\exp(\lambda V_t(\alpha))]$ exists for any $0 < a < \varpi/2B$ and for $t \geq \varphi(2Ba)$. From lemma 4.5,

$$P\left[\frac{1}{t}V_t(\alpha) \geq \frac{\Delta}{3}\right] \leq \frac{M(2Ba,\alpha)-1}{BaE\left[Y_1(\alpha)\right]}\exp\left(-\frac{a\Delta t}{3}\right), \qquad \forall 0 < a < \frac{\varpi}{2B}, \forall t \geq \varphi(2Ba)$$

❑

Remark 6.4. Both remarks 6.1 and 6.2 apply to theorem 6.3.

7. Conclusion

In this paper, we provided some theoretical evidences to support basic principles of ordinal optimization approaches. In particular, we proved the exponential convergence rate of a fundamental indicator called confidence probability: the probability that at least r of the observed top-k designs are the actual top-m designs (i.e. satisfactory designs).We also showed that a kind of positive correlation (i.e. association) improves the convergence rate of the probability that the observed best design is indeed optimal.

Further research includes the investigation of the convergence properties of other stochastic processes and the application of the results of this paper to the design of efficient ordinal optimization approaches. The relationship between the correlation of systems under simulation and the convergence rate is another research direction of interest.

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